August 1995 LMU-TPW 95-10
(revised, Feb. 1996)

Identical Particles and Quantum Symmetries

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Abstract

We propose a solution to the problem of compatibility of Bose-Fermi statistics with symmetry transformations implemented by compact quantum groups of Drinfel'd type. We use unitary transformations to conjugate multiparticle symmetry postulates, so as to obtain a twisted realization of the symmetric groups S_n.

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1 Introduction

Quantum groups [1, 2, 3] have received much attention in recent years as candidates for generalized symmetry transformations in physics. Among other applications, they look promising in relation to generalized space-time[1] and/or internal symmetries in Quantum Field Theory. One way to approach QFT consists first in finding a consistent procedure to implement quantum group transformations in Quantum Mechanics with a finite number of particles, then to pass to QFT through second quantization. Various models describing systems of one particle (see e.g. ref. [5, 6, 7, 8, 9]) or a finite number of *distinct* particles consistently transforming under the action of a quantum group have been constructed so far; as known, the quantum group coproduct plays a specific role in extending quantum group transformations from one-particle to multi-particle systems. In this article we would like to study whether the notions of identical particles and quantum group transformations are compatible in quantum mechanics (in first quantization).

The setting that we have in mind is a quantum mechanical system transforming under generalized (symmetry) transformations realized by some $*$-Hopf algebra H [2] (in particular, a $*$-quantum group [1]). In order that a system of n bosons/fermions transforms under the action of H its Hilbert space of states should carry both a representation of the symmetric group S_n and of H. In the case that the H is quantum group, one might expect that this is impossible.

We recall that in the standard quantum mechanical formalism the elements of S_n are realized as ordinary permutation operators. On the other hand, in the Hopf algebra formalism the action of H on a multiparticle system is defined through the coproduct Δ. Given a representation ρ of H on a "one-particle" Hilbert space $\mathcal{H}$, and considering (for simplicity) the case of two particles, the action of H on $\mathcal{H}\otimes\mathcal{H}$ is defined through $(\rho\otimes\rho)\circ\Delta$. In the case that H is cocommutative (*e.g.* $H = U(su(2))$), the coproduct takes the form $\Delta(X_i) = X_i\otimes 1+1\otimes X_i$ on all the generators X_i (in the case $H = U(su(2))$ this expresses the classical addition law of angular

[1]These are symmetries of a proposed non-commutative structure of space-time [4].

[2]The transformations may correspond to a symmetry either in the sense that they leave the *dynamics* of the particular system under consideration invariant (e.g. rotation symmetry of its Hamiltonian), and therefore are associated to conservation laws for the latter; or in the sense that they leave the *form* of the physical description of *any* system invariant (covariance of the physical description), as it happens e.g. with the Poincaré transformations in Special Relativity.

momentum); therefore the above action preserves the symmetric and antisymmetric subspaces $(\mathcal{H}\otimes\mathcal{H})_\pm$ defined by $P_{12}(\mathcal{H}\otimes\mathcal{H})_\pm = \pm(\mathcal{H}\otimes\mathcal{H})_\pm$ respectively (P_{12} denotes the permutation operator). When H is not cocommutative, *e.g.* it is a quantum group, Δ is no more symmetric under the action of P_{12}, so that the above action mixes $(\mathcal{H}\otimes\mathcal{H})_+$ and $(\mathcal{H}\otimes\mathcal{H})_-$. Therefore, fermions and bosons in the ordinary sense seem impossible, and it is natural to speculate that in the quantum group context some new (or "q-") statistics is necessary or even that the notion of identical particles must be abandoned.

Even if H is just a *slight* deformation of a co-commutative Hopf algebra (e.g. an ordinary Lie group) a new statistics would result into a drastic discontinuity of the number of allowed states of the multi-particle system in the limit of vanishing deformation parameter ($\ln q$ in the $H = U_q\mathbf{g}$-case): in fact, elementary particles cannot be "almost identical", they can only be either identical or different. However, such a discontinuity appears physically unacceptable if we think of H as a slight modification of some experimentally well-established symmetry of elementary particle physics.

A previously suggested "quick fix" of the problem is the naive symmetrization of coproducts —this approach will however destroy any true quantum symmetry. It is also important to realize that it is not enough to make sense of $\Delta(H), \Delta^2(H)$, *etc.* The spaces of multi-particle operators have to be larger than that to be in one-to-one correspondence with their classical (symmetrized) counterparts. In the $H = U_q(su(2))$ case, for instance, we would like to construct the q-analog of the (classically) symmetric operators $X_i\otimes X_i$, which are not the coproduct of anything.

In this work we want to show that a solution to the problem is a modification of our notions of symmetry and anti-symmetry associated to bosons and fermions. The point is that ordinary permutations are not the only possible realization of elements of the abstract group S_n; an alternative one can be obtained by applying some unitary transformation $F_{12\ldots n}$ to the permutators (see section 2). The question (see section 4) is therefore whether for any number of particles n there exists some $F_{12\ldots n}$ (the "twist") such that the corresponding realization of S_n is compatible with the action of H. Due to some theorems by Drinfel'd, this turns out to be the case *at least* if $H = U_q\mathbf{g}$ [1, 2, 3] is one of the standard quantum groups associated to the compact[3] simple Lie algebras $\mathbf{g}$ of the classical series—the case of $U_q\mathrm{su}(2)$ will

[3]For $\mathrm{U}_q(\mathbf{g})$ this requires $q \in \mathbf{R}$. To study the problem in the case of q on the unit circle the

be studied in some detail in section 5—or if H is a triangular Hopf algebra arising from the quantization of a solution of the classical Yang-Baxter equation [11, 12] or from a twist of type [13] as *e.g.* studied in [14][4]. The precise criterion is that H must be the twist of a co-commutative (quasi-)Hopf algebra [16]; in either case we also need the existence of a $*$-conjugation.

In the case where H is a quasi-triangular Hopf algebra one might have expected to see anyons arise as a consequence of the braid group character of $\mathcal{R}$; however, in our formulation this does not happen: The statistics parameter is not modified—bosons stay bosons[5], fermions stay fermions, and anyons (though not studied explicitly) stay anyons. The extreme case of $q = -1$ is especially instructive in this context [19, 20].

Let us ask now how in the context of identical particles the existence of quantum group symmetries of the above kind could manifest itself experimentally: The dynamical evolution of a system of n identical particles will contain new physics only if we adopt an Hamiltonian which is *natural* to the twisted picture. One can always obtain a Hamiltonian *consistent* with twisted symmetrization postulates by a unitary transformation (through $F_{12\ldots n}$) on a Hamiltonian corresponding to some undeformed model, however, such a Hamiltonian will in general be of a very complicated, *i.e.* unnatural form. In section 3 we will analyze a scattering experiment to see how the twist will manifest itself in the transformation of the initial and final data (which is essentially the tensor product of one-particle states) into the equivalent twisted (anti-)symmetrized states upon which the evolution operator describing the scattering acts. The $F_{12\ldots n}$ can again be absorbed in a redefined Hamiltonian, so that an experiment cannot decide whether we are in the twisted picture or not. (It is just a change of base.) We can only tell what picture is more natural. The main message is then that the twisted picture can be consistently introduced. (Contrary to expectation, there are no problems with statistics.) The twisted picture may lead to the development of models that one would probably not think of otherwise. One would expect to see *direct* consequences of the twists only in particle creation and annihilation processes; this however belongs to the realm of quantum field theory and will be treated elsewhere.

reader should consult [10] for the structure of weak quasi-Hopf algebras.

[4]Twisted coproducts are here interpreted as clustered 2-particle states.

[5]See [17, 18] and the extended list of references therein for a discussion of this point in the context of q-deformed oscillators.

For readers not familiar with the notion of Hopf algebras, we give a very brief introduction to the subject in section 6.

After completion of this work we became aware of the very interesting paper in Ref. [21], which gives a quantization scheme for fields transforming covariantly under $SU_q(N)$. In a future work we will compare our results with the ones therein while considering the issue of second quantization.

2 Twisted Multi-Particle Description

Let us forget the issue of quantum symmetry and hence the coproduct for the moment, and just consider pure quantum mechanics for identical particles. Consider a one-particle system, denote by $\mathcal{H}$ the Hilbert space of its states, and by $\mathcal{A}$ the $*$-algebra of observables acting on $\mathcal{H}$. n-particle states and n-particle operators will live in as yet to be determined subspaces of $\mathcal{H}^{\otimes n}$ and $\mathcal{A}^{\otimes n}$ respectively.

Let us consider states of two identical particles. The corresponding state vector $|\psi^{(2)}\rangle$ will be some element of the tensor product of two copies of the one-particle Hilbert space $\mathcal{H}$. Let P_{12} be the permutation operator on $\mathcal{H}\otimes\mathcal{H}$: $P_{12}(|a\rangle\otimes|b\rangle)\equiv|b\rangle\otimes|a\rangle$. (In the sequel we will also use the symbol τ to denote the abstract permutation map of two tensor factors, $\tau(a\otimes b)\equiv b\otimes a$.) The fact that we are dealing with identical particles manifests itself in the properties of state vectors under permutation:

$$P_{12}|\psi^{(2)}\rangle = e^{i\nu}|\psi^{(2)}\rangle, \tag{2.1}$$

where $\nu=0$ for Bose-statistics and $\nu=\pi$ for Fermi-statistics. For the corresponding expectation value of an arbitrary operator $\mathcal{O}\in\mathcal{A}\otimes\mathcal{A}$ we then find

$$\langle\psi^{(2)}|\mathcal{O}|\psi^{(2)}\rangle = \langle\psi^{(2)}|P_{12}\,\mathcal{O}\,P_{12}|\psi^{(2)}\rangle \tag{2.2}$$

because the phases $e^{-i\nu}$ and $e^{i\nu}$ from the bra and the ket cancel. This means that the operators $\mathcal{O}$ and $\tau(\mathcal{O})\equiv P_{12}\mathcal{O}P_{12}$ are members of the same equivalence class as far as expectation values go. One particular representative of each such equivalence class is the symmetrized operator

$$\frac{1}{2}(\mathcal{O}+\tau(\mathcal{O}))\in(\mathcal{A}\otimes\mathcal{A})_+. \tag{2.3}$$

It plays a special role because it preserves the two-particle Hilbert spaces for any statistic (2.1), as we will recall below. We can hence avoid redundant operators by

restricting $\mathcal{A}\otimes\mathcal{A}$ to the sub-algebra

$$(\mathcal{A}\otimes\mathcal{A})_+ := \{a\in\mathcal{A}\otimes\mathcal{A} : [P_{12},a]=0\} \tag{2.4}$$

(note that $[P_{12},a]=0 \Leftrightarrow \tau(a)=a$). In this article we will show how to find an analog of $(\mathcal{A}\otimes\mathcal{A})_+$ compatible with quantum group transformations.

We summarize the relevant equations characterizing a system of two bosons or fermions:

$$P_{12}|u\rangle_\pm = \pm|u\rangle_\pm \qquad \text{for} \quad |u\rangle_\pm \in (\mathcal{H}\otimes\mathcal{H})_\pm \tag{2.5}$$

$$a : (\mathcal{H}\otimes\mathcal{H})_\pm \to (\mathcal{H}\otimes\mathcal{H})_\pm \qquad \text{for} \quad a\in(\mathcal{A}\otimes\mathcal{A})_+ \tag{2.6}$$

$$*_2 : (\mathcal{A}\otimes\mathcal{A})_+ \to (\mathcal{A}\otimes\mathcal{A})_+, \qquad \text{where} \quad *_2 \equiv *\otimes *. \tag{2.7}$$

Equation (2.5) defines bosonic $(+)$ and fermionic $(-)$ states as in (2.1). Equation (2.6) follows from $[P_{12},(\mathcal{A}\otimes\mathcal{A})_+]=0$ and shows that symmetrized operators transform boson states into bosons states and fermion states into fermion states.

Similar statements as given here for two particles obviously apply also to states of 3 and more identical particles and to other statistics (anyons).

Can one also describe in a non-standard way the system of n identical particles, using what we know for one particle, so that the description is perfectly consistent from the physical viewpoint? Let us concentrate on two-particle systems for the moment:

For a unitary and in general not symmetric operator $F_{12}\in\mathcal{A}\otimes\mathcal{A}$, $F_{12}^{*_2}=F_{12}^{-1}$ where $*_2 = *\otimes *$, we define

$$(\mathcal{H}\otimes\mathcal{H})_\pm^{F_{12}} := F_{12}(\mathcal{H}\otimes\mathcal{H})_\pm \tag{2.8}$$

$$P_{12}^{F_{12}} := F_{12}P_{12}F_{12}^{-1} \tag{2.9}$$

$$(\mathcal{A}\otimes\mathcal{A})_+^{F_{12}} := F_{12}(\mathcal{A}\otimes\mathcal{A})_+F_{12}^{-1} \tag{2.10}$$

where $(\mathcal{A}\otimes\mathcal{A})_+$ is as given above. We then find in complete analogy to equations (2.5 – 2.7)

$$P_{12}^{F_{12}}|u\rangle_\pm = \pm|u\rangle_\pm \qquad \text{for} \quad |u\rangle_\pm \in (\mathcal{H}\otimes\mathcal{H})_\pm^{F_{12}} \tag{2.11}$$

$$a : (\mathcal{H}\otimes\mathcal{H})_\pm^{F_{12}} \to (\mathcal{H}\otimes\mathcal{H})_\pm^{F_{12}} \qquad \text{for} \quad a\in(\mathcal{A}\otimes\mathcal{A})_+^{F_{12}} \tag{2.12}$$

$$*_2 : (\mathcal{A}\otimes\mathcal{A})_+^{F_{12}} \to (\mathcal{A}\otimes\mathcal{A})_+^{F_{12}} \tag{2.13}$$

and $a^{F_{12}} := F_{12}aF_{12}^{-1}$ is hermitean iff a is. Equation (2.12) follows from

$$[P_{12}^{F_{12}},(\mathcal{A}\otimes\mathcal{A})_+^{F_{12}}]=0. \tag{2.14}$$

In general, $(\mathcal{H}\otimes\mathcal{H})_\pm^{F_{12}}$ will not be (anti-)symmetric, nor will $(\mathcal{A}\otimes\mathcal{A})_+^{F_{12}}$ be symmetric. Can we still interpret $(\mathcal{H}\otimes\mathcal{H})_\pm^{F_{12}}$ as the Hilbert space of states of the system of two bosons or fermions of equal type and $(\mathcal{A}\otimes\mathcal{A})_+^{F_{12}}$ as the corresponding $*$-algebra of observables? We can. In fact, we have just conjugated the standard description of the 2-particle system through F_{12} into a unitary equivalent one. (This agrees with the general viewpoint put forward in [22, 23]. See also next section for a discussion from the physical point of view.)

Obviously the idea of conjugation can be generalized to a system of n identical particles: Let $F_{12\ldots n} \in \mathcal{A}^{\otimes n}$ be unitary, *i.e.* $(F_{12\ldots n})^{*_n} = (F_{12\ldots n})^{-1}$, where $*_n := *^{\otimes n}$, and define

$$\begin{aligned}
(\mathcal{H}\otimes\ldots\otimes\mathcal{H})_\pm^{F_{12\ldots n}} &:= F_{12\ldots n}(\mathcal{H}\otimes\ldots\otimes\mathcal{H})_\pm && (2.15)\\
P_{12}^{F_{12\ldots n}} &:= F_{12\ldots n}P_{12}(F_{12\ldots n})^{-1} && (2.16)\\
&\vdots\\
P_{n-1,n}^{F_{12\ldots n}} &:= F_{12\ldots n}P_{n-1,n}(F_{12\ldots n})^{-1} && (2.17)\\
(\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+^{F_{12\ldots n}} &:= F_{12\ldots n}(\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+(F_{12\ldots n})^{-1} && (2.18)
\end{aligned}$$

where

$$(\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+ := \{a\in\mathcal{A}\otimes\ldots\otimes\mathcal{A} : [P_{i,i+1},a]=0, i=1,\ldots n-1\},$$

and $P_{i,i+1}$ is the permutator of the $i^{th},(i+1)^{th}$ tensor factors. Then

$$\begin{aligned}
&P_{i,i+1}^{F_{12\ldots n}}|u\rangle_\pm = \pm|u\rangle_\pm \qquad \text{for} \quad |u\rangle_\pm \in (\mathcal{H}\otimes\ldots\otimes\mathcal{H})_\pm^{F_{12\ldots n}} && (2.19)\\
&a : (\mathcal{H}\otimes\ldots\otimes\mathcal{H})_\pm^{F_{12\ldots n}} \to (\mathcal{H}\otimes\ldots\otimes\mathcal{H})_\pm^{F_{12\ldots n}} && (2.20)\\
&\qquad\qquad\qquad\qquad \text{for} \quad a\in(\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+^{F_{12\ldots n}} && (2.21)\\
&*_n : (\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+^{F_{12\ldots n}} \to (\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+^{F_{12\ldots n}}. && (2.22)
\end{aligned}$$

Equation (2.20) follows from

$$[P_{i,i+1}^{F_{12\ldots n}}, (\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+^{F_{12\ldots n}}] = 0. \tag{2.23}$$

Note that in eqs. (2.19) to (2.23) the twist $F_{12\ldots n}$ does not explicitly appear any more; these equations give an *intrinsic* characterization of the twisted multi-particle description, involving only the operators $P_{i,i+1}^{F_{12\ldots n}}$.

By construction $P_{i,i+1}^{F_{12\ldots n}}$ is hermitean, its square is the identity and (consequently) has only eigenvalues ± 1; moreover, the degeneracy of these eigenvalues

is the same as in the untwisted case. The operators $P^{F_{12...n}}_{i,i+1}$ give a realization of the group S_n of permutation of n objects, because they satisfy the same algebraic relations as the ordinary permutators $P_{i,i+1}$; correspondingly, $(\mathcal{A}\otimes\ldots\otimes\mathcal{A})^{F_{12...n}}_{+}$, $(\mathcal{H}\otimes\ldots\otimes\mathcal{H})^{F_{12...n}}_{\pm}$ carries irreducible representations of S_n. Viceversa, one could easily prove that, given operators satisfying these conditions, one can find a unitary $F_{12...n}$ such that equations (2.15) to (2.18) hold.

It will turn out that, even though the twists which are relevant for the quantum symmetry issue are very hard to compute, the $P^{F_{12...n}}_{i,i+1}$ are much less so; see section 5. *Remark:* If we replace the nilpotent P_{12} by some braid group generator one could also conjugacy transform anyons.

3 Identical Versus Distinct Particles

In some situations particles of the same kind can be equivalently treated as *identical* or *distinct*, and there exists a precise correspondence between these two descriptions. The twist F directly enters the rule governing this correspondence while in the twisted postulates (2.19) to (2.23) (intrinsic formulation) it appears only hidden in the P^F (together with its inverse). Transforming one kind of description into the other one is often needed for practical purposes, as we illustrate by the following example.

Consider a gedanken experiment of a scattering of two identical particles. One can distinguish three stages. In the initial stage, the two particles are far apart and are assumed to be prepared in two separate one-particle normalized states $|\psi_1\rangle$, $|\psi_1\rangle$ with vanishing overlap. In the intermediate stage, the particles approach each other and scatter. In the final stage, long after the collision, the particles are again far apart and are detected by one-particle detectors. In the initial and final stage we perform essentially one-particle preparations/measurements, i.e. we have the choice to treat the particles as distinct, whereas in the intermediate stage the collision is correctly described only if we apply a symmetric evolution operator to a properly (anti-)symmetrized two-particle state, that is, if we treat the two particles as identical. The existence of two equivalent descriptions ("distinct" versus "identical") of the two-particle system in the initial and final stages and the *correspondence rule* that relates the two is an essential ingredient of the standard quantum-mechanical formalism.

In this section we want to determine how the conditions for the existence of two equivalent descriptions ("distinct" and "identical") and the correspondence rule between the latter are modified in the twisted formalism. As a by-product, we will realize that closed systems can still be described consistently: If we e.g. want to describe a system of identical particles in our lab we are essentially allowed to forget about the existence of other particles of the same kind in the universe.

Let us consider two-particle scattering again: Let initial states $|\psi_1\rangle, |\psi_2\rangle$ range on some orthogonal subspaces $\mathcal{H}_1, \mathcal{H}_2$ of the whole Hilbert space.[6]

(1) We can treat the two particles as *distinct* particles described by the state

$$|\psi_d\rangle := |\psi_1\rangle \otimes |\psi_2\rangle \qquad \in \qquad \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{3.1}$$

A measurement process is described via a two-particle observable $\mathcal{O}_1 \otimes \mathcal{O}_2$, $\mathcal{O}_i : \mathcal{H}_i \to \mathcal{H}_i$; the probability amplitude to find the two-particle system in a state $|\psi_d'\rangle := |\psi_1'\rangle \otimes |\psi_2'\rangle$ is $\langle\psi_d|\psi_d'\rangle = \langle\psi_2|\psi_2'\rangle\langle\psi_1|\psi_1'\rangle$. This amounts respectively to measuring $\mathcal{O}_1$ on the first *and* $\mathcal{O}_2$ on the second, and to the probability amplitude to find particle 1 in state $|\psi_1'\rangle$ *and* particle 2 in state $|\psi_2'\rangle$. In particular, setting $\mathcal{O}_2 = id$, $|\psi_2'\rangle = |\psi_2\rangle$ means that we neglect the information that we have on the second particle, *i.e.* we ignore its existence.

(2) We can treat them as *identical* particles forming a two-particle system and describe the latter by the twisted (anti)symmetrized state

$$|\psi\rangle = P^{F_{12}}_{S/A}|\psi_d\rangle := \frac{F_{12}}{\sqrt{2}}\left(|\psi_1\rangle \otimes |\psi_2\rangle \pm |\psi_2\rangle \otimes |\psi_1\rangle\right) \in (\mathcal{H} \otimes \mathcal{H})^{F_{12}}_{\pm} \tag{3.2}$$

of bosons (+) or fermions (−). The measurement process of (1) is now described by acting on $|\psi\rangle$ through the twisted symmetrized two-particle observable $F_{12}\left(\mathcal{O}_1 \otimes \mathcal{O}_2 + \mathcal{O}_2 \otimes \mathcal{O}_1\right) F_{12}^{-1} \in (\mathcal{A} \otimes A)^{F_{12}}_{+}$.

Description (2) is perfectly equivalent to (1) because the mapping (1)→(2) preserves scalar products between states (*i.e.* probability amplitudes: $\langle\psi'|\psi\rangle = \langle\psi_d'|\psi_d\rangle = \langle\psi_1'|\psi_1\rangle\langle\psi_2'|\psi_2\rangle$) and spectra of the observables (*i.e.* results of measurements).

For the dynamical evolution, including the collision, it is necessary to use description (2), which involves in an essential way the quantum statistics. Nevertheless, if for later times the state $|\psi\rangle(t)$ becomes a combination of states of the

[6]More generally, if the preparation were uncomplete, we would assume that particle 1,2 is in a *mixture* of states of $\mathcal{H}_1, \mathcal{H}_2$ respectively.

form

$$|\psi\rangle(t) = \sum_{i,j} a_{ij} \frac{F_{12}}{\sqrt{2}} \left(|i\rangle \otimes |j\rangle \pm |j\rangle \otimes |i\rangle\right), \tag{3.3}$$

where $|i\rangle \in \mathcal{H}'_1$, $|j\rangle \in \mathcal{H}'_2$, and $\mathcal{H}'_1$, $\mathcal{H}'_2$ are *orthogonal* subspaces of $\mathcal{H}$ (describing e.g. the states of the particle in detectors 1,2 respectively) then description (1) can be implemented again: we can apply F_{12}^{-1} and drop the (anti-)symmetrization to get the state

$$|\psi_d\rangle(t) = \sum_{i,j} a_{ij} |i\rangle \otimes |j\rangle, \tag{3.4}$$

which will give the final correlation between the potential measurements in the two detectors.

The case of more than two particles can be treated in analogy to the case of two particles. Now however we will want to split the collection of particles into two (or more) *subsystems* instead of into single particles. If there is negligeable overlap between subsystems we are again not forced to treat *all* particles as identical particles; we can describe particles belonging to different subsystems as distinct, but we still have to twist (anti-)symmetrize each subsystem.

If we look at the dynamical evolution, then the same considerations as in the case of two particles will apply. In particular as long as the interaction (of any kind) between a subsystems and the remaining particles is negligeable then we have the choice to consider one subsystem as isolated (implying that we forget the other particles) or of treating all particles as identical.

These considerations hold also when the total number of particles of one kind is very large (virtually infinite) compared to the number in one subsystem. Take this subsystem to be our laboratory and we see that as in the standard formulation, to compute any concrete prediction we can but we don't have to consider all particles of the given type present in the universe at the same time [description "identical"], namely we may ignore the ones "outside our laboratory" [description "distinct"]. In principle however we could apply the postulates of identical particles, through description "identical", to *all* particles of the same type in the universe, without finding inconsistent predictions. In other words, the twisted postulates of Quantum Mechanics for identical particles are completely general and self-consistent.

4 Quantum Symmetries

While their introduction was shown to be consistent, there was so far no need for the $F_{12\ldots n}$. Now we take the issue of quantum group symmetries into consideration.

The picture we have in mind is that of a multi-particle quantum mechanical model (consisting of identical particles) on which we would like to implement generalized (symmetry) transformations through the action of a generic Hopf algebra H.[7] As given data we take the constituent one-particle system, governed by a $*$-algebra $\mathcal{A}$ of operators that act on a Hilbert space $\mathcal{H}$, a $*$-Hopf algebra H with coproduct Δ, counit ε, antipode S and complex conjugation $*$, and a unitary realization ρ of H in $\mathcal{A}$.

The key idea that leads to a construction of multi-particle systems that consistently transform under Hopf algebra actions is that properties of the coproduct should have to do with (twisted) (anti-)symmetry of states and operators. We will find that coproducts should be considered as being (twisted) symmetric—even when we are dealing with non-cocommutative Hopf algebras as symmetries.

Let us start by recalling what it means that a one-particle system transforms under the action of H.

4.1 One-Particle Transformations

To begin, we need a representation ρ of H on $\mathcal{H}$ which realizes H in $\mathcal{A}$:[8]

$$\rho\,:\,H\;\rightarrow\;\mathcal{A}; \tag{4.1}$$

the map ρ is linear and an algebra homomorphism $\rho(xy) = \rho(x)\rho(y)$; $\rho(1_H) = 1_{\mathcal{A}}$ is the identity operator on $\mathcal{H}$. It is called a unitary representation if in addition

$$\rho(x)^* = \rho(x^*). \tag{4.2}$$

(For a representation that is not unitary we would find in contrast $\rho(x)^* = \overline{\rho^\vee}(x^*)$, where $\overline{\rho^\vee}$ is the complex conjugate of the contragredient representation. For a matrix representation: $(T^\vee)^i{}_j = S(T^j{}_i) = (T^{-1})^j{}_i$.)

[7] Later we will concentrate on the case of a twisted image of a cocommutative (quasi-) Hopf algebra; *e.g.* $U_q(g)$.

[8] A given algebra of operators might first have to be extended for this scope.

Let $x \in H$, $\mathcal{O} \in \mathcal{A}$ and $|\psi\rangle \in \mathcal{H}$. The actions of x on the one-particle states $|\psi\rangle$ and and $\mathcal{O}|\psi\rangle$ are given via ρ

$$
\begin{aligned}
x \triangleright |\psi\rangle &= \rho(x)|\psi\rangle, && (4.3)\\
x \triangleright (\mathcal{O}|\psi\rangle) &= \rho(x)\mathcal{O}|\psi\rangle, && (4.4)
\end{aligned}
$$

while on the other hand the action of x on the product $\mathcal{O}|\psi\rangle$ (that is, on an element of the bigger H-module containing both $\mathcal{A}$ and $\mathcal{H}$) should be computed with the coproduct Δ, *i.e.*

$$x \triangleright (\mathcal{O}|\psi\rangle) = (x_{(1)} \overset{s}{\triangleright} \mathcal{O})(x_{(2)} \triangleright |\psi\rangle). \tag{4.5}$$

Here and in the sequel we will use Sweedler's notation $\Delta(x) \equiv x_{(1)} \otimes x_{(2)}$ for the coproduct (in the RHS a sum $\sum_i x^i_{(1)} \otimes x^i_{(2)}$ of many terms is implicitly understood); similarly, $\Delta^{(n-1)}(x) \equiv x_{(1)} \otimes \ldots \otimes x_{(n)}$ for the $(n-1)$-fold coproduct in Sweedler's notation. As known, it follows that the action of H on the one-particle operator $\mathcal{O}$ is given by[9]

$$x \overset{s}{\triangleright} \mathcal{O} = \rho(x_{(1)})\, \mathcal{O}\, \rho(Sx_{(2)}), \qquad x \in H,\ \mathcal{O} \in \mathcal{A}. \tag{4.6}$$

As a concrete example, the reader may think of the case of quantum mechanics in ordinary three-dimensional space with transformations consisting of ordinary rotations; in that case H is the (undeformed) universal enveloping algebra $U(su(2))$ of the (covering of the) Lie group $SO(3)$. ρ maps elements of $U(su(2))$ into operators acting on $\mathcal{H}$, out of which we can single out unitary operators "U" realizing finite rotations (i.e. elements of $SO(3)$), as well as hermitean ones "x" realizing infinitesimal rotations (i.e. elements of $su(2)$) and generating the whole algebra; in these two cases the action (4.6) reduces respectively to conjugation $U\mathcal{O}U^{-1}$ and to taking the commutator $[ix, \mathcal{O}]$. A rotation symmetry of the Hamiltonian usually turns elements of $\rho(U(su(2))$ (*e.g.* angular momentum components) into useful observables for studying the dynamics of the system.

4.1.1 Unitary Transformations

Hermitean conjugation turns an element of $\mathcal{H}$, a "ket", into a "bra" which lives in $\mathcal{H}^*$ and transforms under the contragredient representation. This picture should be preserved under transformations. As we know, in the classical case only unitary and—in the infinitesimal case—anti-hermitean transformation operators have the

[9] See however the remark on page 11.

required property. In the general Hopf algebra case the required property is $S(x) = x^*$; we will call such elements of H *quantum unitary*. We stress the point that there are two notions of unitarity which should not be confused: that of a representation, and that of a transformation. Quantum unitary elements also leave the $*$-structure of $\mathcal{A}$ invariant [24]. The condition for an element $u \in H$ to satisfy

$$(u \overset{s}{\triangleright} \mathcal{O})^* = u \overset{s}{\triangleright} \mathcal{O}^* \qquad \forall \mathcal{O} \in A \tag{4.7}$$

is again

$$u^* = S(u) \qquad \text{(quantum unitary operator).} \tag{4.8}$$

This is seen as follows: $*$-conjugating both sides of equation (4.6) we find a condition

$$\rho(Su_{(2)})^* \otimes \rho(u_{(1)})^* \overset{!}{=} \rho(u_{(1)}) \otimes \rho(Su_{(2)}), \tag{4.9}$$

or, using that ρ is a unitary representation,

$$(Su_{(2)})^* \otimes (u_{(1)})^* \overset{!}{=} u_{(1)} \otimes Su_{(2)}. \tag{4.10}$$

Taking the counit $(\varepsilon \otimes id)$ of this equation gives condition (4.8). A straightforward calculation that uses again unitarity of the representation ρ and standard facts about $*$-Hopf algebras, like $* \circ S = S^{-1} \circ *$ shows that condition (4.8) is in fact sufficient for (4.7).

Remark: There exist pathological Hopf algebras (*e.g.* with $\tau \circ \Delta = (id \otimes S^2)\Delta$) that are not $*$-Hopf algebras but still allow unitary transformations in a non-standard way.

4.2 Multi-Particle Transformations

To implement symmetry transformations (the action of H) on multi-particle systems one makes use of the coproduct of H, which enters the game in two essentially different ways.

First, the coproduct is needed to extend the action of H from one-particle *states* to n-particle states in a way that preserves the twisted (anti)-symmetry of identical particle states. This will constrain the choice of F in section 2, and consequently also the twisted symmetry of operators, according to formula (2.18). On the other hand, the coproduct also enters the action of H on single and multiparticle operators $\mathcal{O}^{(n)}$ [see formula (4.6) for the one-particle case]; if the particles are identical this action

should again preserve the twisted symmetry of the operators. It turns out that both consistency requirements can be simultaneously satisfied through an appropriate choice of the F's.

4.2.1 Transformation of States

We have so far required that $\mathcal{H}$ be a $*$ H-module, *i.e.* that it carries a $*$ representation of H. The main task in constructing Hilbert spaces for identical particles is then to find an operation of twist (anti-) symmetrization that is compatible with the action of H, *i.e.* compatible with the quantum symmetry transformations. The action of H on a multi-particle Hilbert space is given once $\rho^{(n)}$ is known. A representation ρ on the 1-particle Hilbert space extends to a unitary representation on the n-particle Hilbert space via the $(n-1)$-fold coproduct of H:

$$\rho^{(n)} = \rho^{\otimes n} \circ \Delta^{(n-1)} \,:\, H \,\rightarrow\, \mathcal{A}^{\otimes n} \,: \mathcal{H}^{\otimes n} \,\rightarrow \mathcal{H}^{\otimes n}. \tag{4.11}$$

If ρ is unitary then so is $\rho^{(n)}$, $\rho^{(n)}(x)^{*_n} = \rho^{(n)}(x^*)$, because $(* \otimes *) \circ \Delta = \Delta \circ *$.

Let $x \in H$ and $|\psi^{(n)}\rangle \in \mathcal{H}^{\otimes n}$, then

$$x \triangleright |\psi^{(n)}\rangle = \rho^{(n)}(x)|\psi^{(n)}\rangle = \rho(x_{(1)}) \otimes \ldots \otimes \rho(x_{(n)})|\psi^{(n)}\rangle. \tag{4.12}$$

As always we will first consider the case of two particles. Similar considerations will apply to the case of $n \geq 3$ particles. Let P_{12} be the permutation operator on $\mathcal{H} \otimes \mathcal{H}$.

Symmetric coproduct In the case of a *co-commutative* (*i.e.* symmetric under permutation) coproduct we have

$$P_{12}\left((\rho \otimes \rho)\Delta_c(x)\right) = \left((\rho \otimes \rho)\Delta_c(x)\right) P_{12}$$

and hence

$$P_{12}(x \triangleright |\psi^{(2)}\rangle) = x \triangleright (P_{12}|\psi^{(2)}\rangle).$$

This fact allows us to define symmetrizers $P_S = \frac{1}{2}(I + P_{12})$ and anti-symmetrizers $P_A = \frac{1}{2}(I - P_{12})$ that commute with the action of x, and (anti-) symmetrized Hilbert spaces

$$P_S(\mathcal{H} \otimes \mathcal{H}) \;\equiv\; (\mathcal{H} \otimes \mathcal{H})_+, \tag{4.13}$$
$$P_A(\mathcal{H} \otimes \mathcal{H}) \;\equiv\; (\mathcal{H} \otimes \mathcal{H})_-, \tag{4.14}$$

that are invariant under the action of x. This happens for instance if $H = U(\mathbf{g})$, $\mathbf{g} = Lie(G)$. Then $U(\mathbf{g})$ is generated by primitive elements X_i with coproduct

$$\Delta^{(n)}(X_i) = \Delta_c^{(n)}(X_i) = X_i \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes X_i; \tag{4.15}$$

$\Delta_c^{(n)}(X_i)$ is invariant under permutations and we can set $F_{12\ldots n} = \mathbf{1} \otimes \ldots \otimes \mathbf{1}$.

Deformed coproduct If the coproduct is *not co-commutative*, as it happens for a generic Hopf algebra, then the problem arises that the action of H on $(\mathcal{H} \otimes \mathcal{H})$ will no more preserve the subspaces $(\mathcal{H} \otimes \mathcal{H})_\pm$. While we should not change the form of the coproduct (it is at the very heart of quantum groups and tells us how to act on tensor products) we may however modify our notion of symmetric operators and (anti-) symmetrized Hilbert spaces. We can require that

$$\rho^{(n)}(H) \subset (\underbrace{\mathcal{A} \otimes \ldots \otimes \mathcal{A}}_{n\text{-times}})_+^{F_{12\ldots n}} \tag{4.16}$$

for some $F_{12\ldots n}$, so that the system of n identical particles carries a $*$-representation of H as well. This is certainly satisfied if

$$\rho^{(n)}(X) = F_{12..n}\rho_c^{(n)}(X)F_{12..n}^{-1}, \tag{4.17}$$

where $\rho_c^{(n)} := \rho^{\otimes n} \circ \Delta^{(n-1)}$ and Δ_c is a co-commutative coproduct. Equation (4.17) has to be read as a condition on both Δ_c and $F_{12..n}$.

If $H = U_q\mathbf{g}$ [1, 2, 3], where $\mathbf{g}$ is the Lie algebra of one of the simple Lie groups of the classical series, the following theorem due to Drinfel'd and Kohno will be our guidance to the correct choice of the F's we need to satisfy equations (4.16) and (4.17):

Drinfel'd Proposition 3.16 in Ref. [16]

1. *There exists an algebra isomorphism $\phi : U_q\mathbf{g} \leftrightarrow (U\mathbf{g})([[h]])$, where $h = \ln q$ is the deformation parameter.*

2. *If we identify the isomorphic elements of $U_q\mathbf{g}$ and $(U\mathbf{g})([[h]])$ then there exists an $\mathcal{F} \in U_q\mathbf{g} \otimes U_q\mathbf{g}$ such that:*

$$\Delta(a) = \mathcal{F}\Delta_c(a)\mathcal{F}^{-1}, \qquad \forall a \in U_q\mathbf{g} = (U\mathbf{g})([[h]]) \tag{4.18}$$

where Δ is the coproduct of $U_q\mathbf{g}$ and Δ_c is the (co-commutative) coproduct of $U(\mathbf{g})$.

3. *$(U\mathbf{g})([[h]])$ is a quasi-triangular quasi-Hopf algebra (QTQHA) with universal $\mathcal{R}_\Phi = q^{t/2}$ and a quasi-coassociative structure given by an element $\Phi \in ((U\mathbf{g})^{\otimes 3}([[h]]))$ that is expressible in terms of $\mathcal{F}$. $(U\mathbf{g})([[h]])$ as QTQHA can be transformed via the twist by $\mathcal{F}$ into the quasi-triangular Hopf algebra $U_q\mathbf{g}$; in particular, the universal $\mathcal{R}$ of $U_q\mathbf{g}$ is given by $\mathcal{R} = \mathcal{F}_{21}\mathcal{R}_\Phi\mathcal{F}^{-1}$.*

Here $(U\mathbf{g})([[h]])$ denotes the algebra of formal power series in the elements of a basis of $\mathbf{g}$, with coefficients being entire functions of h; $(U\mathbf{g})([[h]])|_{h=0} = U\mathbf{g}$. Point 1) essentially says that it is possible to find h-dependent functions of the generators of $U\mathbf{g}$ which satisfy the algebra relations of the Drinfel'd-Jimbo generators of $U_q\mathbf{g}$ and vice versa.

We recall here that the quasi-triangular Hopf algebras $U_q\mathbf{g}$ can be obtained as quantizations of Poisson-Lie groups associated with solutions of the modified classical Yang-Baxter equations (MCYBE) corresponding to $\mathbf{g}$.

If the Hopf algebra H can be obtained as the quantization of a Poisson-Lie group associated with a solution of the classical Yang-Baxter equation (CYBE) corresponding to some $\mathbf{g}$,[10] then another (and chronologically preceding) theorem by Drinfel'd [11] states the existence of a different $\mathcal{F}$ with similar properties as in the previous theorem—except that now it is enough to twist $(U\mathbf{g})([[h]])$ equipped with the ordinary *coassociative* structure in order to obtain H. The quasi-coassociative structure Φ and the quasi-triangular structure $\mathcal{R}_\Phi$ of point 3) in the theorem reduce to $\Phi = \mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$, $\mathcal{R}_\Phi = \mathbf{1}\otimes\mathbf{1}$; the universal $\mathcal{R}$ is given by $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$. A simple introduction to these topics can be found for instance in Ref. [12].

As shown in Ref. [25], one can always choose a unitary $\mathcal{F}$, if H is a compact section of $U_q(g)$ (*i.e.* when $q \in \mathbf{R}$). If the $\mathcal{F}$ one starts with is not unitary, when one simply multiplies it with the invariant (!) tensor $(\mathcal{F}^*\mathcal{F})^{-1/2}$ to obtain a new $\tilde{\mathcal{F}}$ that is unitary.

These theorems suggest that one can use the unitary twisting operator $\mathcal{F}$ to build F_{12} for a 2-particle sytem. For example:

1. If $\mathcal{A} = \rho(U_q\mathbf{g})$, then we choose

$$F = \rho^{\otimes 2}(\mathcal{F}).$$

[10]In this case H is is triangular, *i.e.* $\mathcal{R}_{21}\mathcal{R}_{12} = \mathbf{1}$

2. If $\mathcal{A}$ = classical Heisenberg algebra $\otimes U^{\mathrm{spin}}(su(2)) \otimes \rho(U_q\mathbf{g})$, were $U_q\mathbf{g}$ plays the role of an internal symmetry, then we can set

$$F_{12} = \mathrm{id}^{(2)}_{\mathrm{Heisenberg}} \otimes \mathrm{id}^{(2)}_{\mathrm{spin}} \otimes \rho^{\otimes 2}(\mathcal{F})$$

3. If $\mathcal{A}$ is the q-deformed Poincaré algebra of ref. [6, 26], and H is the corresponding q-deformed Lorentz Hopf algebra, realized through ρ in $\mathcal{A}$, then we can again define

$$F_{12} = \rho^{\otimes 2}(\mathcal{F}),$$

   where $\mathcal{F}$ belongs to the homogeneous part. The same applies for other inhomogeneous algebras, like the q-Euclidean ones, constructed from the braided semi-direct product [26] of a quantum space and of the corresponding homogeneous quantum group. For both of these examples the one-particle representation theory is known [6, 8].

For n-particle systems one can set $F_{12\ldots n} = \rho^{\otimes n}(\mathcal{F}_{12\ldots n})$, where now we should choose one particular element $\mathcal{F}_{12\ldots n}$ of $H^{\otimes n}$ satisfying the condition

$$\Delta(x) = \mathcal{F}_{12\ldots n}\Delta_c(x)(\mathcal{F}_{12\ldots n})^{-1}. \tag{4.19}$$

To obtain one such $\mathcal{F}_{12\ldots n}$ it is enough to act on eq. (4.18) $(n-2)$ times with the coproduct in some arbitrary order. When $n = 3$, for instance, one can use either $\mathcal{F}'_{123} := [(\Delta\otimes id)(\mathcal{F})]\mathcal{F}_{12}$ or $\mathcal{F}''_{123} := [(id\otimes\Delta)(\mathcal{F})]\mathcal{F}_{23}$. These two elements coincide in the case previously mentioned of Hopf algebras associated to solutions of the CYBE, as proved by Drinfeld [11]. In the the case of $U_q\mathbf{g}$, they do not coincide, but nevertheless $\Phi := \mathcal{F}''_{123}(\mathcal{F}'_{123})^{-1} \neq \mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$ commutes with $\Delta^{(2)}(H)$. In section (5) we will show (in the $U_q(su(2))$ case) how to find a continuous family of $\mathcal{F}_{123}$ interpolating between $\mathcal{F}'_{123}$ and $\mathcal{F}''_{123}$.

Note: From (4.18) follows $(\tau\circ\Delta)(a) = \mathcal{M}\Delta(a)\mathcal{M}^{-1}$ with $\mathcal{M} := \mathcal{F}_{21}\mathcal{F}^{-1}$. This is not the usual relation $(\tau\circ\Delta)(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}$ of a quasi-triangular Hopf algebra; the latter is rather obtained by rewriting equation (4.18) in the form $\Delta(a) = \mathcal{F}q^{t/2}\Delta_c(a)q^{-t/2}\mathcal{F}^{-1}$ where $t = \Delta_c(C_c) - 1\otimes C_c - C_c\otimes 1$ is the invariant tensor ($[t,\Delta_c(a)] = 0 \;\; \forall \;\; a \in U\mathbf{g}$) corresponding to the Killing metric, and C_c is the quadratic casimir of $U\mathbf{g}$. $\mathcal{M}$, unlike $\mathcal{R}$, has not nice properties under the coproducts $\Delta\otimes id$, $id\otimes\Delta$. The reader might wonder whether we could use equation

$[P_{12}R, (\mathcal{A}\otimes\mathcal{A})'_+] = 0$ (where $R = \rho^{\otimes 2}(\mathcal{R})$), instead of eq. (2.14), to single out a modified symmetric algebra $(\mathcal{A}\otimes\mathcal{A})'_+ \subset \mathcal{A}\otimes\mathcal{A}$; in fact, the former is also an equation fulfilled by $\rho^{\otimes 2}(\Delta(H))$ and reduces to the classical eq. (2.4) in the limit $q \to 1$. However $[P_{12}R, (\mathcal{A}\otimes\mathcal{A})'_+] = 0$ is fulfilled *only* by the sub-algebra $\rho^{\otimes 2}(\Delta(H)) \subset (\mathcal{A}\otimes\mathcal{A})$ itself, essentially because $q^{t/2}$ does not commute with *all* symmetric operators, but only with the ones corresponding to coproducts. Therefore, $(\mathcal{A}\otimes\mathcal{A})'_+$ defined via $P_{12}R$ (instead of $P_{12}^{F_{12}}$) is not big enough to be in one-to-one correspondence with the classical $(\mathcal{A}\otimes\mathcal{A})_+$, *i.e.* is not suitable for our purposes.

Explicit universal $\mathcal{F}$'s for $U_q\mathbf{g}$ are not given in the literature, up to our knowledge; an explicit universal $\mathcal{F}$ for a family of deformations (which include quantizations of solutions of both of a CYBE and of a MCBYE) of the Heisenberg group in one dimension was given in Ref. [27].

However, for most practical purposes one has to deal with representations F of $\mathcal{F}$. A general method for constructing the matrices F acting on tensor products of two arbitrary irreducible representations of compact sections of $U_q\mathbf{g}$ is presented in Ref. [22]—there explicit formulas are given for the A, B, C, D-series in the fundamental representation. In [28] matrices twisting the classical coproduct into the q-deformed one were constructed from q-Clebsch-Gordan coefficients.

Moreover, in the intrinsic formulation of the twisted (anti-)symmetrization postulates [eqs. (2.19) – (2.23)] one only needs the twisted permutators $P_{12\ldots n}^{F_{12\ldots n}}$ (not the $F_{12\ldots n}$ themselves); explicit universal expressions for the latter can be found much more easily, as we show in section 5 for $P_{12}^{\mathcal{F}_{12}}$ in the case $H = U_q(su(2))$.

We conclude that the quantum symmetry is compatible with identical particle *states* in the twisted multi-particle description.

4.2.2 Transformation of Operators

Now we want to see if a consistent transformation of the twisted-symmetric operators can be defined.

As we have seen in section 4.1, the action on one-particle operators which makes eq. (4.5) consistent with eq. (4.4) looks formally like the quantum adjoint action. A subtle but important change in the definition of the action on multi-particle operators is needed in order to reach the same goal for multi-particle systems. Our task in this section is twofold: first we have to find the right action of the Hopf algebra H for tensor products of $\mathcal{A}$, then we have to show that the definition of

"twist symmetric" operators (associated to identical particles) is stable under this action. As before, we assume that ρ is a unitary representation that realizes the Hopf algebra H of transformations in $\mathcal{A}$.

Let $\mathcal{O}^{(n)} \in \mathcal{A}^{\otimes n}$ (or a properly symmetrized subspace), $|\psi_n\rangle \in \mathcal{H}^{\otimes n}$ (or a properly (anti)symmetrized subspace); we require, as in the one-particle case,

$$(x_{(1)} \overset{s}{\triangleright} \mathcal{O}^{(n)})(x_{(2)} \triangleright |\psi_n\rangle) = x \triangleright (\mathcal{O}^{(n)}|\psi_n\rangle) = \rho^{(n)}(x)\mathcal{O}^{(n)}|\psi_n\rangle. \tag{4.20}$$

Recalling eq. (4.12) it is easy to see that to satisfy this goal the action (4.6) has to generalize to multi-particle operators in the following way:

$$\begin{aligned} x \overset{s}{\triangleright} \mathcal{O}^{(n)} &= \rho^{(n)}(x_{(1)})\, \mathcal{O}^{(n)}\, \rho^{(n)}(Sx_{(2)}) \\ &= \rho^{\otimes n}(x_{(1)} \otimes \ldots \otimes x_{(n)})\, \mathcal{O}^{(n)}\, \rho^{\otimes n}(Sx_{(2n)} \otimes \ldots \otimes Sx_{(n+1)}). \end{aligned} \tag{4.21}$$

Remark: In the case that $\mathcal{O} = \rho(y)$ with $y \in H$ the action on one-particle operators is nothing but the adjoint action $x \overset{\text{ad}}{\triangleright} y = x_{(1)} y S(x_{(2)})$. The action on multi-particle operators is however different: For instance in the case that $\mathcal{O}^{(2)} = (\rho \otimes \rho)(y_i \otimes y^i)$ with $y_i \otimes y^i \in H \otimes H$ we get

$$x \overset{s}{\triangleright} (y_i \otimes y^i) = x_{(1)} y_i S x_{(4)} \otimes x_{(2)} y^i S x_{(3)}$$

and *not*

$$x \overset{\text{ad}}{\triangleright} (y_i \otimes y^i) = x_{(1)} \overset{\text{ad}}{\triangleright} y_i \otimes x_{(2)} \overset{\text{ad}}{\triangleright} y^i = x_{(1)} y_i S x_{(2)} \otimes x_{(3)} y^i S x_{(4)}$$

as one might have expected. Both actions "$\overset{\text{ad}}{\triangleright}$" and "$\overset{s}{\triangleright}$" coincide for co-commutative coproducts. The former action treats multi-particle operators as tensor products of H-modules, the latter action is related to the natural Hopf algebra structure on $\Delta(H)$ that is given in Sweedler's book [29]. Briefly, Sweedler's argument is the following. For any given number n, $\Delta^{(n-1)}(H)$ can be viewed as a Hopf algebra with a natural coproduct. Now formula (4.6) is applicable for any n—we just have to take care to use the natural Hopf algebra structure for each of the $\Delta^{(n-1)}(H)$.[11]

The notion of unitary multi-particle transformations generalizes to n particles in an obvious way,

$$(u \overset{s}{\triangleright} \mathcal{O}^{(n)})^* = u \overset{s}{\triangleright} (\mathcal{O}^{(n)})^* \qquad \forall \mathcal{O}^{(n)} \in A \tag{4.22}$$

and again is satisfied if $u^* = S(u)$.

We now want to show that the transformation we have found is compatible with the symmetrization of operators in the twisted multi-particle description.

[11] The action "$\overset{s}{\triangleright}$" was also used in Ref. [30] to define covariance properties of tensors in $H^{\otimes n}$

Symmetric coproduct First consider the co-commutative case. Let

$$(\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+ = \{a\in\mathcal{A}\otimes\ldots\otimes\mathcal{A} : [P_{i,i+1},a]=0, i=1,\ldots n-1\}$$

be the completely symmetrized space of n-particle operators. In the case of a co-commutative *i.e. symmetric* coproduct Δ_c any of the permutation operators $P_{i,i+1}$ will commute with the action (4.21):

$$\begin{aligned}\Big[P_{i,i+1}\,,\,(x\overset{s}{\triangleright}\mathcal{O}^{(n)})\Big] &= \Big[P_{i,i+1}\,,\,\rho^{\otimes n}\Big(\Delta_c^{(n-1)}(x_{c(1)})\Big)\,\mathcal{O}^{(n)}\,\rho^{\otimes n}\Big(\Delta_c^{(n-1)}(S_c x_{c(2)})\Big)\Big]\\ &= x\overset{s}{\triangleright}[P_{i,i+1}\,,\,\mathcal{O}^{(n)}], \qquad \text{for } \Delta_c \text{ cocommutative}\,.\end{aligned} \tag{4.23}$$

Here $x_{c(1)}\otimes x_{c(2)}\equiv\Delta_c(x)$ and S_c is the cocommutative antipode.

Deformed coproduct Let $\mathcal{F}_{12\ldots n}\in H^{\otimes n}$ be as in equation (4.19) namely such that $\Delta^{(n-1)}(x)=\mathcal{F}_{12\ldots n}\Delta_c^{(n-1)}(x){\mathcal{F}_{12\ldots n}}^{-1}$ for all $x\in H$. As in the previous section we will use its representation $F_{12\ldots n}\equiv\rho^{\otimes n}(F_{12\ldots n})$ for the similarity transformation of section 2. We note that relation (4.23) also holds with $x_{c(1)}\otimes x_{c(2)}$ and S_c replaced by the non-cocomutative $x_{(1)}\otimes x_{(2)}\equiv\Delta(x)$ and the corresponding antipode S:

$$\begin{aligned}&\Big[P_{i,i+1}\,,\,\rho^{\otimes n}(\Delta_c^{(n-1)}(x_{(1)}))\,\mathcal{O}^{(n)}\,\rho^{\otimes n}(\Delta_c^{(n-1)}(Sx_{(2)}))\Big]\\ &=\rho^{\otimes n}(\Delta_c^{(n-1)}(x_{(1)}))[P_{i,i+1}\,,\,\mathcal{O}^{(n)}]\rho^{\otimes n}(\Delta_c^{(n-1)}(Sx_{(2)})).\end{aligned} \tag{4.24}$$

Conjugating this relation by $F_{12\ldots n}$ we easily find the non-cocommutative analog of equation (4.23), because $\rho^{(n)}(x)=\rho^{\otimes n}(\Delta^{(n-1)}(x))=F_{12\ldots n}\rho^{\otimes n}(\Delta_c^{(n-1)}(x))F^{-1}_{12\ldots n}$:

$$\Big[P^{F_{12\ldots n}}_{i,i+1}\,,\,(x\overset{s}{\triangleright}\mathcal{O}^{(n)})\Big]=x\overset{s}{\triangleright}[P^{F_{12\ldots n}}_{i,i+1}\,,\,\mathcal{O}^{(n)}] \qquad \forall x\in H. \tag{4.25}$$

Consequently, since the LHS vanishes if the RHS does:

$$H\;:\;(\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+^{F_{12\ldots n}}\;\to\;(\mathcal{A}\otimes\ldots\otimes\mathcal{A})_+^{F_{12\ldots n}}. \tag{4.26}$$

The quantum symmetry is hence compatible with identical particle *operators* in the twisted multi-particle description.

Remark: The transformation (4.21) is not the only one compatible with the twisted symmetrization. The important point is that the transformation must be based on $\rho^{(n)}(x)=\rho^{\otimes n}(\Delta^{(n-1)}(x))$. The ordinary commutator $[\rho^{(n)}(x)\,,\,\mathcal{O}^{(n)}]$ also leaves $(\mathcal{A}^{\otimes n})_+^{F_{12\ldots n}}$ invariant, simply because $\rho^{(n)}(x)\in(\mathcal{A}^{\otimes n})_+^{F_{12\ldots n}}$. These two transformations usually coincide in ordinary quantum mechanics. Here they have different

interpretations: Let $h \subset H$ be a sub-algebra of H. The operator $\mathcal{O}^{(n)}$, $n \geq 1$, is symmetric (*i.e.* invariant) under the transformations generated by $x \in h$ if

$$x \overset{s}{\triangleright} \mathcal{O}^{(n)} = \mathcal{O}^{(n)} \epsilon(x); \tag{4.27}$$

it may be simultaneously diagonalizable with elements in h if

$$[\rho^{(n)}(x) \, , \, \mathcal{O}^{(n)}] = 0. \tag{4.28}$$

The two properties coincide if $\Delta(h) \subset h \otimes H$. This can be seen as follows:

$$\begin{aligned} \rho^{(n)}(x)\mathcal{O}^{(n)}|\psi_n\rangle &\overset{(4.12)}{=} x \overset{s}{\triangleright} (\mathcal{O}^{(n)}|\psi_n\rangle) \\ \overset{(4.20)}{=} (x_{(1)} \overset{s}{\triangleright} \mathcal{O}^{(n)})(x_{(2)} \overset{s}{\triangleright} |\psi_n\rangle) &\overset{(4.27)}{=} \varepsilon(x_{(1)})\mathcal{O}^{(n)}(x_{(2)} \overset{s}{\triangleright} |\psi_n\rangle) \\ = \mathcal{O}^{(n)}(x \overset{s}{\triangleright} |\psi_n\rangle) &= \mathcal{O}^{(n)}\rho^{(n)}(x)|\psi_n\rangle \end{aligned} \tag{4.29}$$

for any $|\psi_n\rangle \in \mathcal{H}^{\otimes n}$, so that eq. (4.27) implies eq (4.28); in the same way one proves the converse. The physical relevance of this case is self-evident: if both $\mathcal{O}^{(n)}$ and $\rho^{(\otimes n)}(x)$ are hermitean, then they can be diagonalized simultaneously; if one of the two, say $\rho^{(\otimes n)}(x)$, is not hermitean, given an eigenvector $|\psi_n\rangle$ of $\mathcal{O}^{(n)}$, $\rho^{(\otimes n)}(x)|\psi_n\rangle$ will be another belonging to the same eigenvalue.

5 Explicit Example: $H = U_q(su(2))$

We consider as a simple example of a one-particle quantum mechanical system transforming under a quantum group action the case of a q-deformed rotator, $\mathcal{A} \equiv \rho(H) := \rho[U_q(su(2))]$, with $q \in \mathbf{R}^+$. We determine the twisted symmetry of the systems consisting of $n \geq 2$ particles of the same kind.

5.1 $n = 2$ particles

Let us first assume that the states of the system belong to an irreducible $*$-representation of H, namely $\mathcal{H} \equiv V_j$, where V_j denotes the highest weight representation of $U_q(su(2))$ with highest weight $j = 0, \frac{1}{2}, 1, ...$ It is very instructive to find out what $(\mathcal{H} \otimes \mathcal{H})_{\pm}^{F_{12}}$ and $(\mathcal{A} \otimes \mathcal{A})_{+}^{F_{12}}$ in this example are.

According to point 1. of the Drinfel'd theorem, we can identify $U_q(su(2))$ and $U(su(2))$ as algebras; therefore, V_j can be thought as the representation space of either one. Similarly, $V_j \otimes V_j$ can be considered as the carrier space of a (reducible) representation space of either $U_q(su(2)) \otimes U_q(su(2))$ or $U(su(2)) \otimes U(su(2))$; moreover,

$F_{12}(V_j{\otimes}V_j) = V_j{\otimes}V_j$. Thus, we can decompose it into irreducible components either of $U_q(su(2))$ or $U(su(2))$, the operators on it being defined as $\rho^{(2)}(X) = \rho^{\otimes 2}[\Delta(X)]$ or $\rho_c^{(2)}(X) = \rho^{\otimes 2}[\Delta_c(X)]$ respectively:

$$V_j \otimes V_j = \begin{cases} \bigoplus_{0\le l\le j} \mathcal{V}^q_{2(j-l)} \oplus \bigoplus_{0\le l\le j-\frac{1}{2}} \mathcal{V}^q_{2(j-l)-1} \\ \bigoplus_{0\le l\le j} \mathcal{V}_{2(j-l)} \oplus \bigoplus_{0\le l\le j-\frac{1}{2}} \mathcal{V}_{2(j-l)-1}; \end{cases} \tag{5.1}$$

here $\mathcal{V}^q_J$ (resp. $\mathcal{V}_J$) denotes the irreducible component of $U_q(su(2))$ (resp. $U(su(2))$) with highest weight J. Moreover, from point 2) of the theorem it follows

$$F_{12}\mathcal{V}_J = \mathcal{V}^q_J, \tag{5.2}$$

Let us recall now that the $\mathcal{V}_J$'s have well-defined symmetry w.r.t the permutation, namely $\mathcal{V}_{2j}, \mathcal{V}_{2(j-1)}, \ldots$ are symmetric, $\mathcal{V}_{2j-1}, \mathcal{V}_{2j-3}, \ldots$ are antisymmetric. This follows from the fact that $\rho_c^{(2)}(X)$ and P_{12} commute. Hence

$$\begin{aligned} (V_j \otimes V_j)_+ &= \bigoplus_{0\le l\le j} \mathcal{V}_{2(j-l)} \\ (V_j \otimes V_j)_- &= \bigoplus_{0\le l\le j-\frac{1}{2}} \mathcal{V}_{2(j-l)-1}. \end{aligned} \tag{5.3}$$

From eq.'s (5.2), (5.4) we finally find

$$\begin{aligned} (V_j \otimes V_j)_+^{F_{12}} &:= F_{12}(V_j \otimes V_j)_+ = \bigoplus_{0\le l\le j} \mathcal{V}^q_{2(j-l)} \\ (V_j \otimes V_j)_-^{F_{12}} &:= F_{12}(V_j \otimes V_j)_- = \bigoplus_{0\le l\le j-\frac{1}{2}} \mathcal{V}^q_{2(j-l)-1}. \end{aligned} \tag{5.4}$$

This equation says that the subspaces $\mathcal{V}^q_J \subset V_j \otimes V_j$ have well-defined "twisted symmetry". We can use it to build $(V_j \otimes V_j)^{F_{12}}_{\pm}$ recalling how the representations $\mathcal{V}^q_J$ are obtained. For this scope, we just have to recall the explicit algebra relations and coproduct of the generators $h, X^{\pm}$ of $U_q(su(2))$:

$$\begin{aligned} [h, X^{\pm}] &= \pm 2X^{\pm} & [X^+, X^-] &= \frac{q^h - q^{-h}}{q - q^{-1}} \\ \Delta(h) &= \mathbf{1} \otimes h + h \otimes \mathbf{1} & \Delta(X^{\pm}) &= X^{\pm} \otimes q^{-\frac{h}{2}} + q^{\frac{h}{2}} \otimes X^{\pm}. \end{aligned} \tag{5.5}$$

Let $\{|j, m\rangle\}_{m=-j,1-j,\ldots j}$ be an orthonormal basis of V_j consisting of eigenvectors of $\rho(\frac{h}{2})$ with eigenvalues m. The generators $X^{\pm}$ can be represented in terms of this basis in the following way

$$\rho(X^+)|j, m\rangle = \sqrt{[j + m + 1]_q[j - m]_q}|j, m + 1\rangle, \tag{5.6}$$

$$\rho(X^-)|j, m\rangle = \sqrt{[j - m + 1]_q[j + m]_q}|j, m - 1\rangle, \tag{5.7}$$

where $[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$. As well known, the highest weight vector $\|J,J\rangle \in \mathcal{V}^q_J$—from which the whole representation $\mathcal{V}^q_J$ can be generated by repeated applications of $\rho^{(2)}(X^-)$—is obtained by solving the equation $\rho^{(2)}(X^+)\|J,J\rangle = 0$ for the coefficients a_h of the general ansatz

$$\|J,J\rangle = \sum_{h=\max\{-j,J-j\}}^{\min\{j,J+j\}} a_h |j,h\rangle \otimes |j,J-h\rangle. \tag{5.8}$$

Now we are ready to understand the difference between $(H \otimes H)^{F_{12}}_+$ and its sub-algebra $\rho^{(2)}(H)$:

$$\begin{aligned} \rho^{(2)}(H) &\ni a : \mathcal{V}^q_J \to \mathcal{V}^q_J, & (5.9)\\ (H \otimes H)^{F_{12}}_+ &\ni b : (V_j \otimes V_j)^{F_{12}}_\pm \to (V_j \otimes V_j)^{F_{12}}_\pm. & (5.10)\end{aligned}$$

The elements of $[\rho(H) \otimes \rho(H)]_+ \setminus \rho^{(2)}(H)$ will in general map $\mathcal{V}^q_J$ out of itself, into some $\mathcal{V}^q_{J'}$ with $J' \neq J$.

If $\mathcal{H}$ carries a reducible $*$-representation of H, it will be possible to decompose it into irreducible representations V_j,

$$\mathcal{H} = \bigoplus_{j \in \mathcal{J}} V_j \qquad\qquad \mathcal{J} \subset \mathbf{N}_0/2 := \{0, \frac{1}{2}, 1 \ldots\}; \tag{5.11}$$

then

$$\mathcal{H} \otimes \mathcal{H} = \bigoplus_{j_1, j_2 \in \mathcal{J}} V_{j_1} \otimes V_{j_2}, \tag{5.12}$$

and each $V_{j_1} \otimes V_{j_2}$ itself will be a representation. If $j_1 = j_2$, the considerations above apply. If $j_1 \neq j_2$, the irreducible components $\mathcal{V}^q_J$ ($J = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$) contained in $V_{j_1} \otimes V_{j_2}$ of course *will not* have well-defined symmetry (neither classical nor twisted) under permutations. However, the irreducible components $\tilde{\mathcal{V}}^q_J$ contained in $V_{j_2} \otimes V_{j_1}$ will be characterized by the same set of highest weights J. One can split $\mathcal{V}^q_J \oplus \tilde{\mathcal{V}}^q_J$, and therefore $V_{j_1} \otimes V_{j_2} \oplus V_{j_2} \otimes V_{j_1}$, into the direct sum of one (twisted) symmetric and one (twisted) antisymmetric components

$$[V_{j_1} \otimes V_{j_2} \oplus V_{j_2} \otimes V_{j_1}]^{F_{12}}_\pm = F_{12}\frac{1}{2}[\mathbf{1} \pm P_{12}]\,[V_{j_1} \otimes V_{j_2} \oplus V_{j_2} \otimes V_{j_1}]_\pm \tag{5.13}$$

(the symbol F_{12} has to be dropped in the untwisted case). Let $\{\|J,M\rangle^q_{12}\}_{M=-J,\ldots,J}$ be an orthonormal basis of $\mathcal{V}^q_J$ consisting of eigenvectors of $\rho^{(2)}(h)$ and of $\rho^{(2)}(C_q)$ (C_q denotes the casimir), and let

$$\|J,M\rangle^q_{12} := \sum_{m_1,m_2} \mathcal{C}^{j_1,j_2}_{m_1,m_2}(J,M,q)|j_1,m_1\rangle|j_2,m_2\rangle \tag{5.14}$$

be the explicit decomposition of $\|J,M\rangle_{12}^{q}$ in the tensor product basis of $V_{j_1}\otimes V_{j_2}$. Then the set $\{\|J,M\rangle_{21}^{q}\}_{M=-J,\ldots,J}$ with

$$\|J,M\rangle_{21}^{q}:=\sum_{m_1,m_2}\mathcal{C}_{m_2,m_1}^{j_2,j_1}(J,M,q)|j_2,m_2\rangle|j_1,m_1\rangle, \tag{5.15}$$

will be an orthonormal basis of $\tilde{\mathcal{V}}_J^q$ consisting of eigenvectors of $\rho^{(2)}(h)$ and of the casimir $\rho^{(2)}(C_q)$ with the same eigenvalues. Defining

$$\|J,M\rangle_{\pm}^{q}:=N\left(\|J,M\rangle_{12}^{q}\pm\|J,M\rangle_{21}^{q}\right), \qquad N^{-1}:=\sqrt{2} \tag{5.16}$$

we can easily realize that $\{\|J,M\rangle_{\pm}^{q}\}_{J,M}$ is an orthonormal basis of $(V_{j_1}\otimes V_{j_2}\oplus V_{j_2}\otimes V_{j_1})_{\pm}^{F_{12}}$.

Note that, if $j_1=j_2\equiv j$ and we set $N^{-1}=2$ in formula (5.15), then the vectors $\|J,M\rangle_{+}^{q}$ will make up the sameorthonormal basis of $V_j\otimes V_j$ as before(they will have twisted symmetry $(-1)^{J-2j}$, see the previous case) whereas the vectors $\|J,M\rangle_{-}^{q}$ will vanish.

We are now ready to find, as announced in sections 2, 4, the "universal twisted permutator" $P_{12}^{\mathcal{F}_{12}}$ of $U_q(su(2))$, defined throughthe property that the twisted permutation operator $P_{12}^{F_{12}}$ on any tensor product $V\otimes V$ [V being the carrier space of a representation ρ whatever of $U_q(su(2))$] can be obtained by $P_{12}^{F_{12}}=\rho^{\otimes 2}(P_{12}^{\mathcal{F}_{12}})$.

We decompose $V\otimes V$ as in formula (5.12). The casimir of $U_q(su(2))$

$$C_q=X^-X^++\left(\frac{q^{\frac{h+1}{2}}-q^{\frac{-h-1}{2}}}{q-q^{-1}}\right)^2 \tag{5.17}$$

has eigenvalues $([j+\frac{1}{2}]_q)^2$; in the limit $q\to 1$: $C_q\to C_c+\frac{1}{4}$, where C_c is the usual casimir of $U(su(2))$ with eigenvalues $j(j+1)$. Defining $f(z)$ by

$$\log_q[f(z)]:=\left\{\frac{1}{\ln(q)}\sinh^{-1}\left[\frac{(q-q^{-1})\sqrt{z}}{2}\right]\right\}^2-\frac{1}{4}, \tag{5.18}$$

it is easy to verify that $f(C_q)$ has eigenvalues $q^{j(j+1)}$. Let $\hat{R}:=P_{12}[\rho^{\otimes 2}(\mathcal{R})]$. Recalling the formula $\mathcal{R}=\mathcal{F}_{21}q^{\frac{t}{2}}\mathcal{F}_{12}^{-1}$, we realize that the vectors $\|J,M\rangle_{\pm}^{q}\in(V_{j_1}\otimes V_{j_2}\oplus V_{j_2}\otimes V_{j_1})_{\pm}^{F_{12}}$ $(j_1\neq j_2)$ are eigenvectors of $\rho^{\otimes 2}\left[f(\mathbf{1}\otimes C_q)f(C_q\otimes\mathbf{1})\left[f(\Delta(C_q))\right]^{-1}\right]\hat{R}$ and $P_{12}^{F_{12}}$ with the same eigenvalue ± 1. If $j_1=j_2=j$, the same holds for the vectors $\|J,M\rangle_{+}^{q}$ (which form a basis of $V_j\otimes V_j$). Since this holds for all j_1,j_2 appearing in the decomposition (5.12), and if we let j_1,j_2 range on $\mathcal{J}$ the above vectors make up a basis of $V\otimes V$, then

$$P_{12}^{F_{12}}=f(\mathbf{1}\otimes\rho(C_q))f(\rho(C_q)\otimes\mathbf{1})\left[f\left(\rho^{(2)}(C_q)\right)\right]^{-1}\hat{R} \tag{5.19}$$

on $V \otimes V$. We prefer to rewrite $\hat{R}$ as $\hat{R} = [\rho^{\otimes 2}(\mathcal{R}_{21})]P_{12}$, where $\mathcal{R}_{21} = \tau(\mathcal{R})$ and τ is the abstract permutator. Since this equation holds for an arbitrary representation ρ, we can drop the latter symbol and obtain the

Universal expression for the twisted permutation operator of $U_q(su(2))$:

$$P_{12}^{\mathcal{F}_{12}} = f(\mathbf{1} \otimes C_q)f(C_q \otimes \mathbf{1})\,[f(\Delta(C_q))]^{-1}\,\mathcal{R}_{21} \circ \tau \tag{5.20}$$

We omit here the well-known expression for the universal $\mathcal{R}$ [1].

5.2 $n \geq 3$ particles

When $n \geq 3$, for any given space V the decomposition of $\bigotimes^n V$ into irreducible representations of the permutation group contains components with partial/mixed symmetry, beside the completely symmetric and the completely antisymmetric ones. [12] If $n = 3$, for instance, some components can be diagonalized *either* w.r.t. to P_{12} *or* w.r.t. P_{23} (but not w.r.t. both of them simultaneously). If $n = 4$, all components can be diagonalized simultaneously w.r.t. P_{12} and P_{34}, and some will have mixed symmetry (e.g. will be symmetric in the first pair and antisymmetric in the second, or vice versa). We recall that the explicit knowledge of components with mixed/partial symmetry is required to build $(\mathcal{H}^{\otimes n})_{\pm}$ if the Hilbert space $\mathcal{H}$ of one particle is the tensor product of different spaces, $\mathcal{H} = V \otimes V'$, as in example 2 in subsection 4.2.1.

It is easy to realize that similar statements hold in the case of the twisted symmetry.

Let us consider again the case V_j, and let $n = 3$ for the sake of simplicity. We show how to construct two different orthonormal bases of $V_j \otimes V_j \otimes V_j$ with (partial) symmetry, and a continuous family of F_{123} on $V_j \otimes V_j \otimes V_j$.

There is evidently only one irreducible representation with highest weight $J = 3j$, the highest weight vector being $|j,j\rangle|j,j\rangle|j,j\rangle$. But there are two independent irreducible representations with highest weight $J = 3j-1$, e.g. those having highest weight vectors $\frac{1}{\sqrt{2}}\left(|j,j-1\rangle|j,j\rangle \pm |j,j\rangle|j,j-1\rangle\right)|j,j\rangle$. The latter are symmetric and antisymmetric respectively w.r.t. P_{12}, but are mixed into each other by the action of P_{23}; alternatively, one can combine these two representations into two new ones, having highest weight vectors $\frac{1}{\sqrt{2}}|j,j\rangle\left(|j,j-1\rangle|j,j\rangle \pm |j,j\rangle|j,j-1\rangle\right)$, which

[12] The Young tableaus provide the rules for finding the complete decomposition for any n.

are symmetric and antisymmetric respectively w.r.t. P_{23}, but are mixed into each other by the action of P_{12}. One can easily verify that the first two representations are eigenspaces of $\rho_c^{(2)}(C_c)\otimes id$ with eigenvalues $(2j\pm\frac{1}{2})^2$, the latter two are eigenspaces of $id\otimes\rho_c^{(2)}(C_c)$ with the same eigenvalues. The operators $\rho_c^{(3)}(C_c), \rho_c^{(3)}(h)$ and either $\rho_c^{(2)}(C_c)\otimes id$ or $id\otimes\rho_c^{(2)}(C_c)$ make up a complete set of commuting observables over $V_j\otimes V_j\otimes V_j$. Let

$$\{\|J,M,r\rangle_{12}\}_{J,M,r}, \qquad \text{[respectively:} \quad \{\|J,M,s\rangle_{23}\}_{J,M,s}], \tag{5.21}$$

with

$$j\le J\le 3j, \qquad -J\le M\le J, \qquad \max\{0,j-J\}\le r,s\le \min\{2j,j+J\} \tag{5.22}$$

denote an orthonormal basis of eigenvectors of $\rho_c^{(3)}(C_c)$, $\rho^{(3)}(h)$ and $\rho_c^{(2)}(C_c)\otimes id$ [respectively: $id\otimes\rho_c^{(2)}(C_c)$] with eigenvalues $J(J+1), M$ and $r(r+1)$ [respectively: $s(s+1)$]. In particular,

$$\begin{aligned}\|3j-1,3j-1,2j-\frac{1}{2}\pm\frac{1}{2}\rangle_{12} &= \frac{1}{\sqrt{2}}\left(|j,j-1\rangle|j,j\rangle\pm|j,j\rangle|j,j-1\rangle\right)|j,j\rangle\\ \|3j-1,3j-1,2j-\frac{1}{2}\pm\frac{1}{2}\rangle_{23} &= \frac{1}{\sqrt{2}}|j,j\rangle\left(|j,j-1\rangle|j,j\rangle\pm|j,j\rangle|j,j-1\rangle\right)\end{aligned} \tag{5.23}$$

It is easy to verify that in general the subspace of $V_j\otimes V_j\otimes V_j$ which is antisymmetric/symmetric w.r.t. P_{12} is spanned by the vectors $\|J,M,r\rangle_{12}$ with $r-\min\{2j,j+J\}$ odd/even, and similarly for P_{23}.

For fixed J,M, there exists a unitary matrix $U(J)$ such that

$$\|J,M,s\rangle_{23}=U(J)_{sr}\|J,M,r\rangle_{12} \tag{5.24}$$

Formulae formally identical to eqs. (5.22), (5.24) hold when $q\neq 1$; we will introduce an additional index q in all objects to denote this dependence.

The elements of $\rho^{(3)}(U_q(su(2)))$ in these two bases read

$$\rho^{(3)}(X)=\begin{cases}\sum\limits_J\sum\limits_r\sum\limits_{M,M'}X_{M,M'}(J)\|J,M,r,q\rangle_{12}\ {}_{12}\langle J,M,r,q\|\\ \sum\limits_J\sum\limits_s\sum\limits_{M,M'}X_{M,M'}(J)\|J,M,s,q\rangle_{23}\ {}_{23}\langle J,M,s,q\|,\end{cases} \tag{5.25}$$

and the matrix elements $X_{M,M'}(J)$ do not depend on r,s.

Now it is easy to check that we can find many-parameter continuous families of matrices F_{123} satisfying eq. (4.19), in the form

$$F_{123}=\begin{cases}\sum\limits_J\sum\limits_M\sum\limits_r A_{r,r'}(J)\|J,M,r,q\rangle_{12}\ {}_{12}\langle J,M,r',1\rangle\|\\ \sum\limits_J\sum\limits_M\sum\limits_s B_{s,s'}(J)\|J,M,s,q\rangle_{23}\ {}_{23}\langle J,M,s',1\|,\end{cases} \tag{5.26}$$

where $A(J)$'s are arbitrary unitary matrices and $B(J) = [U(J,q)]^* A(J) [U(J,q = 1)]^T$. The key point is that the matrix elements $A_{r,r'}$ do not depend on M, whereas the matrix elements $X_{M,M'}$ do not depend on r.

It is easy to realize that the family (5.26) interpolates between the two F matrix given in subsection 4.2.1, F'_{123} (if we set $A_{r,r'} = \delta_{r,r'}$) and F''_{123} (if we set $B_{r,r'} = \delta_{r,r'}$).

Considerations analogous to those of subsection 5.1 can be done for $n \geq 3$ when V is a reducible representation of $U_q(su(2))$.

Acknowledgments

We would like to thank R. Engeldinger for sharing his interpretation of Drinfel'd twists which helped us in a change of viewpoint on the problem and we are grateful to S. L. Woronowicz and W. Pusz for pointing out Ref. [21]. One of us (G.F.) thanks A. v. Humboldt foundation for financial support. Last but not least we thank J. Wess for discussion and warm hospitality at his institute.

References

[1] V. G. Drinfeld, *Quantum Groups*, Proceedings of the International Congress of Mathematicians, Berkeley 1986, Vol. 1, 798.

[2] M. Jimbo, Lett. Math. Phys. **10** (1986), 63.

[3] L. D. Faddeev, N. Y. Reshetikhin and L. A. Takhtajan, *Quantization of Lie Groups and Lie Algebras*, Algebra i Analysis, **1** (1989), 178; translation: Leningrad Math. J. **1** (1990), 193.

[4] O. Ogievetsky, W. B. Schmidke, J. Wess and B. Zumino, Commun. Math. Phys. **150** (1992), 495.

[5] J. Wess, B. Zumino, *Differential Calculus on Quantum Planes and Applications*, Talk given on the occasion of the Third Centenary Celebrations of the Mathematische Gesellschaft Hamburg, March 1990, KA-THEP-1990-22.

[6] M. Pillin, W. B. Schmidke and J. Wess, Nucl. Phys. **B403** (1993), 223.

[7] G. Fiore, Int. J. Mod. Phys. **A 8** (1993), 4679; *$SO_q(N)$-isotropic Harmonic Oscillator on the Quantum Euclidean space $\mathbf{R}_q^N$*, LMU-TPW 94-26 and q-alg/9412011.

[8] G. Fiore, *The Euclidean Hopf algebra $U_q(e^N)$ and its fundamental Hilbert space representations*, J. Math. Phys. in press; *The q-Euclidean algebra $U_q(e^N)$ and the corresponding q-Euclidean lattice*, LMU-TPW 95-4 and q-alg/9505028.

[9] W. Weich, *The Hilbert Space Representations for $SO_q(3)$ Symmetric Quantum Mechanics*, LMU-TPW-1994-5 and hep-th/9404029.

[10] G. Mack, V. Schomerus, Commun. Math. Phys. **149** (1992), 513.

[11] V. G. Drinfeld, Doklady AN SSSR **273** (1983), 531.

[12] L. A. Takhtajan, *Lectures on Quantum Groups*, in "Introduction to Quantum Groups and Integrable Massive Models of Quantum Field Theory", Nankai Lectures on Mathematical Physics, (World Scientific, 1990).

[13] N. Y. Reshetikhin, Lett. Math. Phys. **20** (1990), 331.

[14] J. Lukierski, H. Ruegg, V. N. Tolstoy and A. Nowicki, *Twisted Classical Poincaré Algebras*, J. Phys. **A 27** (1994), 2389.

[15] P. Podles, S. L. Woronowicz, *On the Classification of Quantum Poincaré Groups*, HEPTH-9412059 and hep-th/9412059; *On the structure of Inhomogeneous Quantum Groups*, HEPTH-9412058 and hep-th/9412058.

[16] V. G. Drinfeld, *Quasi Hopf Algebras*, Leningrad Math. J. **1** (1990), 1419.

[17] S. Vokos, C. Zachos, *Thermodynamic q-distributions that aren't*, Mod. Phys. Lett. **A 9** (1994), 1.

[18] C. A. Nelson, M. H. Fields, Phys. Rev. **A 51** (1995), 2410.

[19] C. Zachos, Mod. Phys. Lett. **A7** (1992), 1559.

[20] R. Zhang, Lett. Math. Phys. **25** (1992), 317.

[21] W. Pusz, S. L. Woronowicz, *Twisted Second Quantization*, Reports on Mathematical Physics **27** (1989), 231.

[22] R. Engeldinger, *On the Drinfel'd-Kohno Equivalence of Groups and Quantum Groups*, Preprint LMU-TPW 95-13 and q-alg/9509001.

[23] R. Engeldinger, unpublished note (1994) and private communication.

[24] P. Schupp, Ph.D. thesis, UC Berkeley (1993); UMI-94-30673-mc (micro fiche), LBL-34942 and hep-th/9312075.

[25] B. Jurco, Commun. Math. Phys. **166** (1994), 63.

[26] S. Majid, J. Math. Phys. **34** (1993), 2045.

[27] F. Bonechi, R. Giachetti, E. Sorace, M. Tarlini, *Deformation Quantization of the Heisenberg Group*, Commun. Math. Phys. **169** (1995), 627.

[28] T. L. Curtright, G. I. Ghandour, C. K. Zachos, J. Math. Phys. **32** (1991), 676.

[29] M. E. Sweedler, *Hopf Algebras*, (Benjamin, New York, 1969).

[30] P. Schupp, P. Watts, and B. Zumino, *Bicovariant Quantum Algebras and Quantum Lie Algebras*, Commun. Math. Phys. **157** (1993), 305.

6 Appendix

The following is a short summary of Hopf algebra notions *relevant to the present article.* For more detailed discussions of these topics the reader could *e.g.* consult [29, 1, 16].

Hopf algebras can be seen as an abstraction from group algebras and (universal enveloping algebras of) Lie groups. Taking part of the representation theory into their very definition, Hopf algebras achieve the unification of such diverse concepts.

Mathematically, a Hopf algebra H is an algebra $H(\cdot, +, k)$ over a field k (typically the field of complex numbers) with additional operations Δ, ϵ, S called the coproduct, counit and antipode respectively, satisfying suitable axioms.
The **coproduct** Δ is an algebra homomorphism, $\Delta : H \to H \otimes H$, that fixes the way representations are combined: Let ρ, ρ' be representations of H on some vector space. The tensor product $\rho \times \rho'$ of these representations is

$$\rho \times \rho' = (\rho \otimes \rho') \circ \Delta. \tag{6.1}$$

Let $l \in \mathbf{g} \subset U\mathbf{g}$ be a Lie algebra element; its coproduct is nothing but the well-known angular-momentum addition rule

$$\Delta(l) = l \otimes 1 + 1 \otimes l. \tag{6.2}$$

(In the case of quantum groups coproducts are in general not symmetric (*i.e.* not co-commutative), see (5.5).) To take tensor products of more than two representations we have to apply the coproduct repeatedly. It does not matter which tensor factors get split again hereby, since the coproduct is co-associative

$$(\Delta \otimes id)\Delta(x) = (id \otimes \Delta)\Delta(x), \qquad \forall x \in H. \tag{6.3}$$

This property is nicely expressed in Sweedlers notation for the coproduct: $(\forall x \in H)$

$$\Delta(x) \equiv x_{(1)} \otimes x_{(2)} \in H \otimes H \tag{6.4}$$

$$(\Delta \otimes id)\Delta(x) \equiv x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \equiv (id \otimes \Delta)\Delta(x) \in H \otimes H \otimes H \tag{6.5}$$

$$\vdots$$

Note that (formal) sums of terms are implied in this notation. Expression (6.1) can now be written as:

$$(\rho \times \rho')(x) = \rho(x_{(1)}) \otimes \rho'(x_{(2)}). \tag{6.6}$$

The **counit** ϵ gives the trivial (1-dimensional) representation:

$$\epsilon: \ H \rightarrow k, \qquad \epsilon(xy) = \epsilon(x)\epsilon(y). \tag{6.7}$$

It holds that

$$\cdot\,(\epsilon \otimes id)\Delta(x) = x = \cdot(id \otimes \epsilon)\Delta(x), \qquad \forall x \in H, \tag{6.8}$$

i.e. $\epsilon \times \rho = \rho = \rho \times \epsilon$ for all representations ρ. We have $\epsilon(1) = 1$, because $\Delta(1) = 1 \otimes 1$, and $\epsilon(l) = 0$ for $l \in \mathbf{g}$ in example (6.2).

The **antipode** S is an anti-algebra map

$$S: \ H \rightarrow H, \qquad S(xy) = S(y)S(x), \tag{6.9}$$

that defines the contragredient representation $\rho^\vee$ to any given representation ρ: $\rho^\vee = \rho^T \circ S$ (^{T} is the transpose). You may think of S as a generalized inverse. It holds that

$$S(x_{(1)})x_{(2)} = 1 \cdot \epsilon(x) = x_{(1)}S(x_{(2)}), \qquad \forall x \in H, \tag{6.10}$$

i.e. $(\rho^\vee)^T \times \rho = \epsilon = \rho \times (\rho^\vee)^T$, and

$$\Delta(S(x)) = S(x_{(2)}) \otimes S(x_{(1)}), \qquad \epsilon(S(x)) = \epsilon(x), \qquad \forall x \in H. \tag{6.11}$$

We have $S(1) = 1$ and $S(l) = -l$ for $l \in \mathbf{g}$ in example (6.2).

Quasi-triangular Hopf-algebras are Hopf-algebras where the non-cocommutativity is under control by a "universal" $\mathcal{R} \in H \otimes H$ such that

$$x_{(2)} \otimes x_{(1)} = \mathcal{R}(x_{(1)} \otimes x_{(2)})\mathcal{R}^{-1}, \qquad \forall x \in H, \tag{6.12}$$

and

$$\Delta_1\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad \Delta_2\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{6.13}$$

(The subscripts indicate tensor factors here—*i.e.* $\Delta_1 \equiv \Delta \otimes id$ and $\mathcal{R}_{13}\mathcal{R}_{23} = \sum_{i,j} a_i \otimes a_j \otimes b_i b_j \in H \otimes H \otimes H$, where $\mathcal{R} \equiv \sum_i a_i \otimes b_i$, *etc.*) It is left as an exercise to the reader to show that (6.12) and (6.13) imply the so-called quantum Yang-Baxter-Equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{6.14}$$

In section 4 we used the concept of quasi-Hopf algebras. These are in general non-coassociative Hopf algebras with an element $\Phi \in H \otimes H \otimes H$ such that

$$(id \otimes \Delta)\Delta(x) = \Phi(\Delta \otimes id)\Delta(x)\Phi^{-1}, \quad \forall x \in H. \tag{6.15}$$